British Journal of Multidisciplinary and Advanced Studies:
Mathematics, Statistics, Quantitative and Operations Research 4(6),1-22, 2023
Print ISSN: 2517-276X
Online ISSN: 2517-2778
Website: https://bjmas.org/index.php/bjmas/index
Published by the European Centre for Research Training and Development UK

Rational Points on Fermat's Surfaces in Minkowski's(N+1) -Dimensional Spaces and Extended Fermat's Last Theorem: Mathematical Framework and Computational Results

Ramon Carbó-Dorca[1,2]

Institut de Química Computacional – Universitat de Girona – Girona 17005 (Catalonia) Spain[1]
Ronin Institute – 127 Haddon Place – Montclair N. J. 07043 USA[2]

doi: https://doi.org/10.37745/bjmas.2022.0353 Published November 26, 2023

Citation: Carbó-Dorca R. (2023) Rational Points on Fermat's Surfaces in Minkowski's(N+1) -Dimensional Spaces and Extended Fermat's Last Theorem: Mathematical Framework and Computational Results, *British Journal of Multidisciplinary and Advanced Studies*: Mathematics, Statistics, Quantitative and Operations Research 4(6),1-22

ABSTRACT: *The Fermat last theorem, defined in (2+1)-dimensional Minkowski spaces, is discussed and extended in natural and rational Mikowski's spaces. Several pieces of computational interest are given, with many practical examples. A definition of Fermat vector order, Fermat surfaces, and Fermat surface radius is given. Several conjectures are discussed, among them the existence of a Fermat theorem in (3+1)-dimensional Minkowski spaces.*

KEYWORDS: Fermat's Last Theorem, fermat surfaces, Minkowski spaces, extended fermat vectors, discrete probability distributions, order and radius of a fermat vector, lattices

INTRODUCTION

Despite the elaborate Wiles demonstration [1], Fermat's last theorem still attracts researchers to this aspect of number theory. For example, recent papers on the subject [2-4] still present simple alternative demonstrations of the theorem. Following this research line, one of us has published several papers [5-8] trying to find extensions of Fermat's theorem.

1 Institut de Química Computacional – Universitat de Girona – Girona 17005 (Catalonia) Spain
ramoncarbodorca@gmail.com
2 Ronin Institute – 127 Haddon Place – Montclair N. J. 07043 USA

Mathematics, Statistics, Quantitative and Operations Research 4(6),1-22, 2023

Print ISSN: 2517-276X

Online ISSN: 2517-2778

Website: https://bjmas.org/index.php/bjmas/index

Published by the European Centre for Research Training and Development UK

These last studies mainly discuss the extension of the theorem to larger dimensions and present it as a heavy-computational problem. Based on the previous empirical work, a naïve demonstration of Fermat's last theorem has recently been published [9]. Such a demonstration has been developed in a vector space framework with a Minkowski structure.

All this previous experience has opened the way for the present study, which has been structured in two parts. The first one describes the Minkowski natural space basis, where the discussion of Fermat's theorem and the extensions to higher dimensional spaces is also provided. In the second, many examples of *N*-dimensional vectors obeying a Fermat-like rule for various powers are presented and discussed.

A general Fermat theorem seems difficult to enunciate, except for a (3+1)-dimensional Minkowski space, where exhaustive computational tests have been performed [7]. One can conclude that the behavior of (3+1)-dimensional vectors appears similar to the original Fermat theorem in a (2+1)-dimensional Minkowski space. In higher-dimensional spaces, one can observe the existence of vectors with properties similar to the two- and three-dimensional ones.

The most interesting algebraic structure corresponds to the connection of the natural vectors subject to a Minkowski metric with rational vectors, which also behave as Fermat vectors with zero Minkowski norms. Such vectors constitute a set of elements that also can be associated with discrete probability distributions.

The existence of Fermat surfaces is also evidenced when the order of the powers is larger than 2. At the same time, spheres and hyperspheres describe the natural Fermat vectors of order 2.

Therefore, this first part of the study of the Fermat theorem and its extensions has been organized in the following way. Vector semispaces are described first, followed by the possible ways to construct them via the inward product of two vectors. From here, one can obtain a set of vectors that behave as scalars. Whole perfect vectors are introduced next, and Minkowski norms are easily defined. Finally, the Fermat vectors of diverse dimensions are studied, starting with the (2+1) and ending in dimensions as large as possible.

1. Vector Semispaces.

An *N*-dimensional semispace (or orthant) defined over the set of positive rational numbers: $\mathbb{Q}^+$, can be symbolized by: $V_N\left(\mathbb{Q}^+\right)$. A semispace [10-18] can be generated by the vectors of a vector space $V_N\left(\mathbb{Q}\right)$ using the following transformation:

$$\begin{aligned}&\forall\langle\mathbf{x}|=(x_1,x_2,x_3,\dots,x_N)\in V_N(\mathbb{Q})\\&\quad\rightarrow\langle|\mathbf{x}||=(|x_1|,|x_2|,|x_3|,\dots,|x_N|)\in V_N(\mathbb{Q}^+)\end{aligned}. \qquad (1)$$

One can extend this definition to vector spaces over the real and complex fields without problems other than using the corresponding field elements in the vector components and the redefinition of the vector $\langle|\mathbf{x}||$ in the case of the complex field; one can use a vector module, possessing modules of complex numbers as elements.

The rational field is used here because of the semispace vector's final computational construction and manipulation.

3. An alternative construction of semispaces: inward product of two vectors.

3.1. Definition of the inward product of two vectors.

Any vector space can be transformed into a set with an extra structure by defining a not-very-much-used product of two vectors, which here will be called the inward product [18]. However, it is well known as a product with other names like Hadamard, Schur, diagonal….

The inward product of two vectors of a vector space $V_N(\mathbb{Q})$ is easily defined as:

$$\forall\{\langle\mathbf{x}|;\langle\mathbf{y}|\}\subset V_N(\mathbb{Q}):\langle\mathbf{p}|=\langle\mathbf{x}|*\langle\mathbf{y}|\in V_N(\mathbb{Q})\Leftarrow\forall I=1,N:p_I=x_I y_I. \qquad (2)$$

Therefore, the inward product of two vectors results in another vector belonging to the same space. Properties of the inward product of two vectors are those of the product of two scalars. Thus, a vector space with the additional definition of the inward product might be transformed into a set whose axiomatic structure *resembles* a field.

3.2. The inward inverse of a vector.

One needs to define an ad hoc property of the inward inverse of a vector, though, leaving the possible zero components null. That is, one can describe the following algorithm:

$$\begin{aligned}&\forall\langle\mathbf{x}|\in V_N(\mathbb{Q}):\\&\quad\exists\langle\mathbf{x}^{[-1]}|\in V_N(\mathbb{Q})\Leftarrow\forall I=1,N:x_I^{[-1]}=\delta(x_I\neq 0)x_I^{-1}+\delta(x_I=0)x_I\end{aligned} \qquad (3)$$

where the symbols $\delta(Expression)$ are logical Kronecker deltas [19], which take the values: $\{0,1\}$, according to the result of the $Expression$ becomes:

Print ISSN: 2517-276X
Online ISSN: 2517-2778
Website: https://bjmas.org/index.php/bjmas/index
Published by the European Centre for Research Training and Development UK

$\{\delta(.False.),\delta(.True.)\}$. For example, the usual Kronecker's delta might be written from this point of view as: $\forall I,J=1,N:\delta_{IJ}\equiv\delta(I=J)$.

3.3. The inward unity vectors.

The definition of inward inverses has to be accompanied by the definition of unity-appropriate vectors, which correspond to the vertices taken with rational elements of the N-dimensional Boolean hypercubes [15]. This definition appears because vectors with zero elements belong to subspaces of lesser dimension than the original vector space. Such subspace vectors can be avoided by defining whole vectors lacking zero components, as developed in section **4.**.

The N-dimensional unity vector: $\langle\mathbf{1}_N| = (1,1,1,\ldots,1)\in V_N(\mathbb{R})$, corresponds to the neutral element for the inward product of vectors in general, and for the whole vector kind (see next section **4.**), one can write:

$$\langle\mathbf{x}|*\langle\mathbf{x}^{[-1]}| = \langle\mathbf{x}^{[-1]}|*\langle\mathbf{x}| = \langle\mathbf{1}_N|. \qquad (4)$$

3.4. The inward power of a vector.

The inward product can be, without exception, extended to any number of vectors and, as such, can be the source of inward natural powers of a vector, as follows:

$$\forall p\in\mathbb{N}\wedge\forall\langle\mathbf{x}|\in V_N(\mathbb{R}):$$
$$\langle\mathbf{x}^{[p]}| = \mathop{*}_{I=1}^{p}\langle\mathbf{x}| = \langle\mathbf{x}|*\langle\mathbf{x}|*\ldots*\langle\mathbf{x}| = (x_1^p, x_2^p, x_3^p,\ldots,x_N^p) \qquad (5)$$

4. Whole perfect vectors.

A (row)[3] vector:

$$\langle\mathbf{w}| = (w_1,w_2,w_3,\ldots,w_N)\in V_N(\mathbb{R}^+) \qquad (6)$$

might be called *whole* (lacking zero components) and *perfect* (whole-perfect or WP) when its elements fulfill the following inequalities:

$$0<w_1<w_2<w_3,\ldots<w_N; \qquad (7)$$

[3] One uses row vectors here, considering that all the vector properties of rows are the same on the dual column vector spaces. The transformation of one form into another can be made by transposition.

that is, they are non-zero, positive, all different, and well-ordered.

The set of whole-perfect vectors can be constructed as a subset of the semispace $V_N(\mathbb{R}^+)$, and symbolized by: $W_N(\mathbb{R}^+) \subset V_N(\mathbb{R}^+)$. The WP subset: $W_N(\mathbb{R}^+)$, is a vector semisubspace, as it is trivial to realize that the sum of WP vectors is a WP vector and a homothecy of a WP vector is a WP vector:

$$\begin{aligned} &\forall \langle \mathbf{w}| \in W_N(\mathbb{R}^+) \wedge \forall \lambda \in \mathbb{R}^+ : \langle \mathbf{h}| = \lambda \langle \mathbf{w}| \rightarrow \langle \mathbf{h}| \in W_N(\mathbb{R}^+) \\ &\forall \{\langle \mathbf{w}_a|, \langle \mathbf{w}_b|\} \subset W_N(\mathbb{R}^+) : \langle \mathbf{s}| = \langle \mathbf{w}_a| + \langle \mathbf{w}_b| \rightarrow \langle \mathbf{s}| \in W_N(\mathbb{R}^+) \end{aligned} \qquad (8)$$

Also, the permutations of the elements of a WP vector generate a set of $N!$ whole vectors. One can construct a basis set of $W_N(\mathbb{R}^+)$ by choosing a subset of N elements among these vector permutations. Thus, a WP vector's circular permutations can be considered a basis set generator.

5. Whole-perfect vectors as homothecies.

In a semispace $V_{N+1}(\mathbb{R}^+)$, one can consider any WP vector constructed as:

$$\forall \langle \mathbf{v}| \in W_{N+1}(\mathbb{R}^+) : \langle \mathbf{v}| = (\langle \mathbf{w}|, r), \qquad (9)$$

where r is the largest vector element: v_{N+1} , which can be called the *radius* of the WP vector.

One can also suppose such a vector form as homothetic to a WP vector with the most significant element transformed into the unity:

$$\begin{aligned} &\forall \langle \mathbf{v}| = (\langle \mathbf{w}|, r) \in W_{N+1}(\mathbb{R}^+) : \langle \mathbf{v}| = r\left(\frac{w_1}{r}; \frac{w_2}{r}; \frac{w_3}{r}; \ldots; \frac{w_N}{r}; 1\right) = r\langle \mathbf{z}| \\ &\Rightarrow \langle \mathbf{z}| = (u_1, u_2, u_3, \ldots u_N, 1) = (\langle \mathbf{u}|, 1) \in Z_{N+1}(\mathbb{R}^+) \wedge \langle \mathbf{u}| \in U_N(1^+) \end{aligned} \qquad (10)$$

where the symbol: $1^+ \equiv (0,1) \subset \mathbb{R}^+$, stands for the open unit interval, and by the symbol $U_N(1^+)$ one can understand the set of N-dimensional WP vectors defined over 1^+. Therefore, a one-to-one correspondence occurs between the elements of both sets

British Journal of Multidisciplinary and Advanced Studies:
Mathematics, Statistics, Quantitative and Operations Research 4(6),1-22, 2023
Print ISSN: 2517-276X
Online ISSN: 2517-2778
Website: https://bjmas.org/index.php/bjmas/index
Published by the European Centre for Research Training and Development UK

$W_{N+1}\left(\mathbb{Q}^{+}\right)$ and $Z_{N+1}\left(\mathbb{Q}^{+}\right)$. One can, therefore, consider the set $W_{N+1}\left(\mathbb{Q}^{+}\right)$ as generated by all the rational positive homotheties of the vectors of $Z_{N+1}\left(\mathbb{Q}^{+}\right)$.

6. Minkowski norms (of order p) over $V_N\left(\mathbb{Q}^{+}\right)$.

One can define a Minkowski norm of order p over the semispace $V_{N+1}\left(\mathbb{Q}^{+}\right)$ as follows:

$$\forall\langle\mathbf{v}|\in V_{N+1}\left(\mathbb{Q}^{+}\right): M_p\left[\langle\mathbf{v}|\right]=\sum_{I=1}^{N} w_I^p - r^p . \qquad (11)$$

If there exists a vector such that:

$$\langle\mathbf{f}|\in V_{N+1}\left(\mathbb{Q}^{+}\right): M_p\left[\langle\mathbf{f}|\right]=0; \qquad (12)$$

then, the vector $\langle\mathbf{f}|$ can be called an *extended* (or rational) Fermat vector (of order p).

The natural vector semispaces or lattices: $V_{N+1}\left(\mathbb{N}\right)$, could be supposed to be subsemiespaces of $V_{N+1}\left(\mathbb{Q}^{+}\right)$, in this case, one can write: $V_{N+1}\left(\mathbb{N}\right)\subset V_{N+1}\left(\mathbb{Q}^{+}\right)$.

The WP natural vectors $\langle\mathbf{t}|$ bearing null Minkowski norms of order p, can be called *true* (or natural) Fermat vectors (of order p), that is, when fulfilling:

$$\langle\mathbf{t}|\in V_N\left(\mathbb{N}\right): M_p\left[\langle\mathbf{t}|\right]=0 . \qquad (13)$$

In these cases, one can also write:

$$\exists\langle\mathbf{t}|=\left(t_1;t_2;t_3;\ldots;t_N;r\right): \qquad (14)$$
$$M_p\left[\langle\mathbf{t}|\right]=\sum_{I=1}^{N} t_I^p - r^p = 0 \rightarrow \sum_{I=1}^{N} t_I^p = r^p$$

meaning that true Fermat vectors are the elements of some Fermat surface of *order p* and *radius r* (see section 8.4. below). $(N+1)$-dimensional true Fermat vectors of order 2 correspond to natural WP vectors sitting on the surface of an N-dimensional hypersphere of radius r.

7. Alternative definition of Minkowski norms (of order p).

7.1. The complete sum of a vector.

The Minkowski norms discussed and defined in the above section can also be constructed by providing a simple linear operator transforming any vector to a scalar. The operator, which can be called the *complete sum* of a vector, can be associated with the following computational algorithm:

$$\forall\langle\mathbf{x}|\in V_N(\mathbb{R}):\langle\langle\mathbf{x}|\rangle=\sum_{I=1}^{N}x_I\in\mathbb{R}\ . \qquad (15)$$

The linearity of the complete sum of a vector is easy to prove and will not be given, as discussed earlier [15-17]. The complete sum of a natural power p of a vector corresponds to the Euclidean norm of order p of the associated vector; thus, one can write in the semispace $V_N(\mathbb{R}^+)$:

$$\forall p\in\mathbb{N}\wedge\forall\langle\mathbf{x}|\in V_N(\mathbb{R}^+):E_p(\langle\mathbf{x}|)=\sum_{I=1}^{N}x_I^p\equiv\langle\langle\mathbf{x}^{[p]}|\rangle. \qquad (16)$$

7.2. The metric vector.

As described earlier, the Minkowski norm of a vector can be easily defined using the *metric* vector [17,18]. The metric vector for Euclidean norms in *N*-dimensional space coincides with the unity vector $\langle\mathbf{1}_N|=(1,1,1,\ldots,1)$.

7.3. The alternative expression of the Minkowski norm of a vector.

Associated with the Minkowski norm of order p as previously defined, the metric vector becomes:

$$\langle\mathbf{m}|=(\langle\mathbf{1}_N|;-1)=(1,1,\ldots,1,-1)\in V_{N+1}(\mathbb{R}), \qquad (17)$$

then one can write for the semispace $V_{N+1}(\mathbb{R}^+)$:

$$\forall p\in\mathbb{N}\wedge\forall\langle\mathbf{x}|\in V_{N+1}(\mathbb{R}^+):$$
$$M_p(\langle\mathbf{x}|)=\langle\langle\mathbf{x}^{[p]}|*\langle\mathbf{m}|\rangle=\langle\langle\langle\mathbf{x}_N^{[p]}|;-x_{N+1}^p|\rangle. \qquad (18)$$

Such a rearrangement of the Minkowski norm is interesting because it is possible that the negative term of the definition, associated in the present paper with the Fermat vectors, can be generalized to any number of negative terms and positions in the ordered vector components. This Minkowski spaces generalization aspect has been previously studied [8,11,13,17,18] and will not be repeated here.

8. The unit hypersphere, discrete probability distributions, and Fermat vectors and surfaces.

8.1. The unit hyperspheres

Any vector space $V_N(\square)$ holds a vector subset $S_N(\square) \subset V_N(\square)$, which can be named the unit hypersphere. One might assign to the vectors of such a subset the property:

$$\forall \langle \mathbf{s} | \in S_N(\square) : \langle \langle \mathbf{s}^{[2]} | \rangle = 1 \leftarrow \sum_{I=1}^{N} s_I^2 = 1, \tag{19}$$

in this manner, the elements of $S_N(\square)$ constitute the surface of an *N*-dimensional hypersphere of unit radius.

8.2. The generators of unit hyperspheres

As one can deduce from the equation, vectors with unit complete sum and squared elements generate the unit hyperspheres.

Also, one can observe the set $S_N(\square)$ as containing two types of generators.

First, considering the squared vectors: $\langle \mathbf{s}^{[2]} | \in S_N^{[2]}(1^+)$, as they have a unit complete sum, they can also be taken as the vector structure of all the discrete probability distributions of a given dimension, just realizing that one can write:

$$\begin{gathered}\forall \langle \mathbf{s} | \in S_N(\square) \rightarrow \langle \langle \mathbf{s}^{[2]} | \rangle = 1 \rightarrow \langle \mathbf{s}^{[2]} | = \langle \mathbf{p} | \in V_N(1^+) \\ \Rightarrow \forall I = 1, N : s_I^2 = p_I \Rightarrow \langle \langle \mathbf{p} | \rangle = 1 \Rightarrow \sum_{I=1}^{N} p_I = 1 \\ \Rightarrow \langle \mathbf{p} | \in P_N(1^+) \subset V_N(\square^+)\end{gathered} \tag{20}$$

Then, the subset of vectors $P_N(1^+)$ contains all the N-dimensional discrete probability distributions, and one can consider it equivalent to the subset holding all the squared vectors defining the hypersphere surface:

British Journal of Multidisciplinary and Advanced Studies:
Mathematics, Statistics, Quantitative and Operations Research 4(6),1-22, 2023
Print ISSN: 2517-276X
Online ISSN: 2517-2778
Website: https://bjmas.org/index.php/bjmas/index
Published by the European Centre for Research Training and Development UK

$$P_N\left(1^+\right) \equiv S_N^{[2]}\left(1^+\right). \tag{21}$$

8.3. Squared probability vectors.

Also, one can use the fact that the property associated to the squared-probability vectors, which can be written as:

$$\begin{aligned} &\forall \langle \mathbf{p}| \in P_N\left(1^+\right) \rightarrow \langle\langle \mathbf{p}|;1| \in V_{N+1}\left(\square^+\right) \\ &\rightarrow \langle\langle \mathbf{s}^{[2]}|;1| \in V_{N+1}\left(\square^+\right) \rightarrow M_2\left(\langle\langle \mathbf{s}^{[2]}|;1|\right) = M_2\left(\langle \mathbf{f}|\right) = 0 \end{aligned} \tag{22}$$

Such expression shows that extended Fermat vectors and discrete probability distributions are related. True Fermat vectors are thus related to a subset of the probability distributions, with all probability terms possessing a well-defined rational structure.

The second generators correspond to the elements of the hypersphere surface set $S_N\left(\square\right)$. They can be used via homotheties as generators of the inner and outer sphere vectors, the homothetic parameter being $\lambda < 1$ in case of generating inner sphere vectors and $\lambda > 1$ for the outer sphere ones.

8.4. Fermat surfaces

Therefore, these self-evident definitions and hypersphere characteristics are of interest in the context of generalized Fermat vectors. In the next lines, one will develop the framework of the characterization of Fermat surfaces, which include surfaces as *N*-dimensional hyperspheres as a particular case.

To properly define a Fermat surface of unit radius, one must consider a set of vectors associated with some vector space, defined over the real or the rational (for computational and practical purposes) field. Therefore, it is necessary to specify the dimension of such a chosen Minkowski vector space, let us say (N+1). A second parameter needed corresponds to the order of the surface, accepting p as the order value.

Then, one can define an *N*-dimensional Fermat surface of order p and radius $r = 1$, as a set of (N+1)-dimensional vectors with p-th order unit norm. Therefore $S_N\left(\square\right)$, the *N*-dimensional hyperspheres of unit radius can be considered *N*-dimensional Fermat surfaces of order 2 and radius $r = 1$.

Thus, any vector space $V_{N+1}\left(\square\right)$ can hold a vector subset $F_N^p\left(\square\right) \subset V_{(N+1)}\left(\square\right)$, which can be defined in the following form:

British Journal of Multidisciplinary and Advanced Studies:
Mathematics, Statistics, Quantitative and Operations Research 4(6),1-22, 2023
Print ISSN: 2517-276X
Online ISSN: 2517-2778
Website: https://bjmas.org/index.php/bjmas/index
Published by the European Centre for Research Training and Development UK

$$\forall \langle \mathbf{f} | = (f_1, f_2, f_3, \ldots, f_N, 1) \in F_N^p(\mathbb{N}) \rightarrow M_p(\langle \mathbf{f} |) = \sum_{I=1}^{N} |f_I|^p - 1 = 0, \qquad (23)$$

the absolute value appears here to consider possible negative vector component values.

Even if one defines the vector space over the complex field, the symbol can be accepted as a module of the vector elements. Then, with the definition of the equation (23), one can consider that: $F_N^2(\mathbb{N}) = S_N(\mathbb{N})$.

Of course, homotheties of the vectors belonging to a Fermat surface define surfaces within the same dimensions but with the radius transformed into the homothecy parameter.

True Fermat vectors whose elements and radius correspond to natural numbers might exist within a homothetic Fermat surface. Still, one cannot ensure that this situation appears as general property.

With the definition of Fermat surfaces, one can be sure that there are *no* natural Fermat vectors in Fermat surfaces, such as $p > 2 : F_2^p(\mathbb{N})$ due to Fermat's last theorem and $p > 3 : F_3^p(\mathbb{N})$ as a conjecture obtained empirically [7, 8].

One must be aware that the sets of vectors associated with constructing Fermat surfaces, when inward powered to the order p of the surface, are related to discrete probability distributions of the adequate dimension, as one has discussed earlier in section 8.3.

Another question is the lack of computed natural Fermat vectors associated with Fermat surfaces when the dimension, the order, or both tend to be large numbers. That is the existence of such surfaces when dimension N and order p tend to be infinite. The problem is complex enough to leave it pending for further research.

9. True Fermat vectors of second-order.

This section might describe trivial extensions of Fermat's last theorem, keeping the order 2 as in the theorem but analyzing the possibility of Natural Fermat vectors of larger dimensions than (2+1).

In this case and the following ones, all the search of true Fermat vectors has been performed, avoiding the natural unit 1 acting as an element of the generated vectors.

9.1. Empirical evidence of true Fermat $(N+1)$-dimensional vectors of second-order.

There is empirical evidence that true Fermat vectors of order 2 exist up to large values of the dimension of natural spaces. An example is the Leech lattice [20-22], and another corresponds to the published computational results [6-8] dealing with natural Fermat vectors up to dimensions $N \le 600$.

Another numerical example[4] can illustrate this ubiquity of Fermat vectors of order 2 in any $(N+1)$-dimensional Minkowski space.

It has been obtained by randomly searching the vector components in the natural number range $\{2, 2^{10}-1\}$ and within a natural space dimension (52+1). The following vector corresponds to one arbitrarily chosen among the set of computed elements:

[15, 43, 53, 111, 113, 122, 130, 135, 141, 157, 160, 174, 186, 257, 261, 264, 267, 269, 293, 357, 360, 438, 497, 537, 556, 560, 569, 576, 588, 614, 615, 639, 658, 670, 697, 726, 739, 751, 752, 785, 810, 812, 823, 836, 846, 859, 889, 926, 931, 978, 1003, 1023] $[r = 4275]$.

The order 2 property is easy to demonstrate for the above vector, just writing a natural WP vector in such terms as previously done and assigning the last element to a factual radius of a hypersphere.

Thus, one can write the second-order Minkowski norm of $(N+1)$-dimensional true Fermat vectors as:

$$M_2\left[\langle \mathbf{t}|\right] = \sum_{I=1}^{N} t_I^2 - r^2 = 0 \Rightarrow \sum_{I=1}^{N} t_I^2 = r^2, \tag{24}$$

after a trivial rearrangement, they yield the equation of the natural points sitting on the surface of an N-dimensional hypersphere of radius r.

Then true Fermat $(N+1)$-dimensional vectors of order 2 are particular points on the surface of a hypersphere of radius r, described in a vector space $V_{N+1}(\square)$, or a more restricted case in the semispace $V_{N+1}(\square^{+})$.

9.2. A conjecture on true Fermat vectors of order 2.

As previously discussed, such a situation indicates one can enunciate a conjecture about: "the existence of an infinite number of true Fermat vectors of order 2 in *any* dimension".

9.3. (2+1)-dimensional true Fermat vectors of order 2.

[4] This and other results in this paper have been obtained under a dedicated Python program, available to the author on demand.

Print ISSN: 2517-276X
Online ISSN: 2517-2778
Website: https://bjmas.org/index.php/bjmas/index
Published by the European Centre for Research Training and Development UK

The multidimensional true Fermat vectors of order 2 generalize the well-known Pythagorean triples, which, according to the present definitions and algebraic construction, correspond to true Fermat (2+1)-dimensional vectors of order 2. Besides, they are directly connected to Fermat's last theorem.

In this case, one has empirically shown that an extension of the last Fermat theorem applies [5,6]. In this three-dimensional Minkowski environment, there are no Fermat vectors of any order different than the trivial first and second-order ones [9].

This property above corresponds to a partial demonstration of Fermat's last theorem. An alternative demonstration to the one proposed by Wiles needs to prove that:

$$\begin{aligned}&\forall\langle\mathbf{v}| = (x,y,r) \in V_{(2+1)}(\square)\\ &\rightarrow \forall p \in \square \wedge p > 2 : M_p(\langle\mathbf{v}|) \neq 0 \leftarrow x^p + y^p - r^p \neq 0\end{aligned} \quad (25)$$

Such property has been demonstrated for the true Fermat vectors in this Minkowski space environment [9], that is:

$$\begin{aligned}&\forall\langle\mathbf{f}| = (a,b,r) \in V_{(2+1)}(\square) \rightarrow M_2(\langle\mathbf{f}|) = 0 \leftarrow a^2 + b^2 - r^2 = 0\\ &\wedge \forall p \in \square \wedge p \neq 2 : M_p(\langle\mathbf{f}|) \neq 0 \leftarrow a^p + b^p - r^p \neq 0\end{aligned} \quad (26)$$

The first part of the equation (26) becomes the same as writing the two-dimensional expression of the equation (10), as one can describe, in this case, the following expression:

$$\forall\langle\mathbf{f}| = (a,b,r) \in V_{(2+1)}(\square) \rightarrow a^2 + b^2 = r^2 \rightarrow \left(\frac{a}{r}\right)^2 + \left(\frac{b}{r}\right)^2 = 1 \quad (27)$$

and the final equation (27) is equivalent to the sine and cosine relationship:

$$\frac{a}{r} = \sin(\alpha) = S \wedge \frac{b}{r} = \cos(\alpha) = C \rightarrow S^2 + C^2 = 1, \quad (28)$$

therefore, for appropriate angle values, one can also write:

$$\forall p \in \square \wedge p \neq 2 : S^p + C^p \neq 1 \rightarrow M_p(\langle\mathbf{f}|) \neq 0, \quad (29)$$

thus, the equation (29) follows Fermat's last theorem, as there will be angle values for which a correct homothecy produces a true Fermat vector.

One must note that extended and true Fermat squared vectors might easily correspond to the collection of rational two-dimensional probability distributions, using the findings of previous section **8.**.

British Journal of Multidisciplinary and Advanced Studies:
Mathematics, Statistics, Quantitative and Operations Research 4(6),1-22, 2023
Print ISSN: 2517-276X
Online ISSN: 2517-2778
Website: https://bjmas.org/index.php/bjmas/index
Published by the European Centre for Research Training and Development UK

9.4. Higher order Minkowski norms.

There is another problem when the natural (2+1)-dimensional vectors for which equalities like the ones in the equation (27) are not fulfilled.

In case they are, though, one can try a reduction to the absurd, writing:

$$\begin{aligned}
&\exists q \in \Box \wedge q \neq 2 \wedge \langle \mathbf{v}| \in V_{(2+1)}(\Box): M_q(\langle \mathbf{v}|) = 0 \\
&\rightarrow a^q + b^q - r^q = 0 \rightarrow \left(\frac{a}{r}\right)^q + \left(\frac{b}{r}\right)^q = 1 \qquad (30) \\
&\rightarrow x = \frac{a}{r} \in \Box^+ \wedge y = \frac{b}{r} \in \Box^+
\end{aligned}$$

That is, it will appear that a relationship similar to the true Fermat vectors will appear with a power value other than 2.

It is a matter of obtaining a contradiction from this relationship, as shown in the equation (30). The relationship will mean that a Fermat's surface of order q can contain at least one rational point: a natural homothetic parameter transforms into a true Fermat vector. The point $\langle \mathbf{p}| = (x, y, 1)$ might be seen as written in cartesian coordinates. Still, as such, it can also be written in terms of the sine and cosine of an appropriate angle and radius unit, so at the same time as the discussed true Fermat vectors, one can write an equivalent vector in terms of sine and cosine, or: $\langle \mathbf{v}| = (S, C, 1)$. Then, the vector $\langle \mathbf{p}|$ can be written in terms of the circular coordinates as:

$$\langle \mathbf{v}^{[q]}| = (S^q, C^q, 1): M_q(\langle \mathbf{v}^{[q]}|) = 0 \rightarrow S^q + C^q = 1, \qquad (31)$$

but if $q \neq 2$ this is not possible, one enters into a contradiction.

Therefore, it seems that there cannot be Fermat vectors other than order 2 in natural Minkowski (2+1)-dimensional spaces.

10. (3+1)-dimensional true Fermat vectors of arbitrary orders.

Empirical evidence based on exhaustive computational results [7] suggests that (3+1)-dimensional true Fermat vectors of order 3 exist likewise to those in the lower (2+1) dimension. Similarly, there is no collected computational evidence of true Fermat vectors of order larger than 3 in (3+1)-dimensional lattices [7,8].

British Journal of Multidisciplinary and Advanced Studies:
Mathematics, Statistics, Quantitative and Operations Research 4(6),1-22, 2023
Print ISSN: 2517-276X
Online ISSN: 2517-2778
Website: https://bjmas.org/index.php/bjmas/index
Published by the European Centre for Research Training and Development UK

However, true Fermat $(N+1)$-dimensional vectors of order 3 exist parallelly as the order 2 vectors exist. In this sense, a conjecture similar to the last Fermat theorem can be enunciated for order 3 Fermat vectors. That is: "an infinite number of true Fermat vectors of order 3 exists in *any* dimension $\geq (3+1)$"

10.1. Fermat vectors of order 3 within large dimensional natural semispaces.

As an example of this, one can write the following vector, obtained in a similar way that in the second-order case, which is a third-order true Fermat vector in a (43+1)-dimensional natural Minkowski semispace:

[30, 32, 112, 120, 132, 158, 207, 213, 220, 233, 266, 312, 374, 375, 387, 397, 415, 442, 472, 517, 549, 573, 580, 584, 588, 607, 610, 632, 644, 650, 664, 670, 676, 707, 741, 774, 815, 828, 830, 878, 889, 891, 951] $[r = 2131]$

Another third-order Fermat vector example follows:

[18, 21, 49, 55, 114, 119, 123, 147, 173, 189, 199, 221, 225, 232, 248, 255] $[r = 456]$

,

this time extracted from a (16+1)-dimensional natural Minkowski semispace. Also, such large dimensions permit the existence of large order vectors, too, like in this example of a true Fermat vector in the dimension (16+1) but order 5:

[12, 26, 28, 42, 51, 53, 71, 73, 79, 84, 95, 99, 112, 115, 121, 124] $[r = 165]$.

10.2. A conjecture of extended Fermat theorem of order 3 in (3+1)-dimensional natural semispaces.

For (3+1)-dimensional natural vectors, Fermat vectors of order greater than 3 seem that don't exist, at least computationally. Meanwhile, for (*N*+1)-dimensional natural vectors, Fermat vectors of order 3 seem to exist without dimension limit.

Then, one can write the following expression:

$$\forall \langle \mathbf{f}| = (a,b,c,r) \in V_{(3+1)}(\mathbb{N}) \rightarrow M_3(\langle \mathbf{f}|) = 0 \leftarrow a^3 + b^3 + c^3 - r^3 = 0$$
$$\wedge \forall p \in \mathbb{N} \wedge p \neq 3 : M_p(\langle \mathbf{f}|) \neq 0 \leftarrow a^p + b^p - r^p \neq 0$$

as a way to describe a Fermat's last theorem-like conjecture within three-dimensional Minkowski spaces and third-order norms.

That is, one can conjecture that: "in (3+1)-dimensional natural Minkowski semispaces, there exist true Fermat vectors of the third order, but no higher-order ones".

10.3. True Fermat vectors of order 3 in higher dimensional spaces.

Also, one can conjecture that third-order true Fermat vectors exist in all (N+1)-dimensional spaces.

For example, a random search, similar to the one used in second-order Fermat vectors, provides many true (6+1)-dimensional Fermat vectors of the third order. Some arbitrarily chosen three specimens are:

[80, 119, 121, 144, 160, 166] $\left[r=250\right]$

[14, 75, 124, 145, 197, 247] $\left[r=304\right]$

[3, 7, 153, 177, 212, 229] $\left[r=313\right]$

Some true Fermat vectors can possess the *same* radius, like the following two vectors obtained with a radius $\left[r=328\right]$. The meaning of this corresponds to considering that one has found two natural Fermat vectors lying on the corresponding Fermat surface of order 3 and radius 328:

[44, 105, 127, 202, 222, 234]

[105, 115, 179, 185, 208, 226].

More examples of existing natural points in Fermat surfaces will be provided below.

11. (4+1)-dimensional and higher-dimensional true Fermat vectors of arbitrary orders.

Empirical evidence points to true Fermat vectors of order 4 and 5 in (4+1)- and (5+1)-dimensional natural Minkowski semispaces. In (4+1)-dimensional natural semispaces, true Fermat vectors of order 4 are present with some scarcity, and occasionally it has even been found some order 5 vectors.

11.1. (4+1)-dimensional true Fermat vectors.

There follows a sample of true Fermat vectors of various orders, associated with a (4+1)-dimensional natural Minkowski semispace:

Order 3: [10, 15, 40, 65] $\left[r=70\right]$

[71, 73, 228, 234] $\left[r=294\right]$

Print ISSN: 2517-276X
Online ISSN: 2517-2778
Website: https://bjmas.org/index.php/bjmas/index
Published by the European Centre for Research Training and Development UK

Order 4: [60, 240, 544, 630] $\left[r=706\right]$

Order 5: [27, 84, 110, 133] $\left[r=144\right]$

[54, 168, 220, 266] $\left[r=289\right]$.

Curiously enough, (4+1)-dimensional true Fermat vectors can be of orders 2, 3, 4, *and* 5. The 4-order vectors are very scarce, and surprisingly, also not so scarce 5-order vectors are present.

Such a characteristic impedes allowing the generalization of the Fermat last theorem to any dimension.

No vectors of higher orders have been computed, but this is no real proof of their lack of existence.

Perhaps they are very scarce and thus hard to generate among many natural number combinations.

However, in larger (N+1) dimensions, the collection of true Fermat vectors of orders larger than N are either scarce or nonexistent, as they have not been computationally found among extensive random searches.

So, perhaps the (4+1)-dimensional case is connected to a very peculiar Fermat surface possessing properties different from other dimensions. But this cannot even be conjectured.

11.2. (5+1)-dimensional true Fermat vectors

The following examples of different (5+1)-dimensional vectors illustrate the results one gets in this kind of dimension. Order 2 vectors are quite numerous and not shown, but one can describe the particular higher-order ones as follows:

Order 3: [8, 40, 51, 66, 77] $\left[r=98\right]$

[28, 78, 94, 98, 117] $\left[r=157\right]$

Order 4: [13, 16, 22, 38, 84] $\left[r=85\right]$

[13, 98, 104, 212, 228] $\left[r=265\right]$

[59, 78, 132, 174, 246]

Order 5: [21, 23, 37, 79, 84] $\left[r=94\right]$.

An interesting feature concerning order 4 has to be noted. There are three Fermat vectors written, one with a radius of 85 and two with the same radius of 265, meaning they correspond to *two different* points of the *same* Fermat surface $F_5^4\left(\mathbb{N}^+\right)$, embedded in a Minkowski semispace of dimension (5+1), order 4, and radius 265.

More examples of Fermat surfaces with many natural points found are given in the following sections.

Orders larger than 5 have not been found in this dimension, so the extended 5-order originated in the (4+1)-dimensional case, which now in dimension (5+1) could be associated with vectors of order 6, seems nonexistent or quite difficult to find.

11.3. Large dimensions

By large dimensions, one refers to dimensions larger than (5+1). Some examples follow.

11.3.1. Dimension (6+1)

Some examples of order 3 have been shown before in section *10.3*. Here, apart from more vectors, also order 4 and 5 are presented:

Order 3: [30, 39, 188, 207, 208, 231] $\left[r=333\right]$

[26, 97, 138, 161, 169, 242] $\left[r=299\right]$

[57, 77, 85, 103, 209, 248]

Order 4: [16, 64, 156, 157, 212, 234] $\left[r=281\right]$

Order 5: [6, 17, 60, 64, 73, 89] $\left[r=99\right]$

In order 3, there are two vectors with the same radius of 299. Both are natural points of the same Fermat surface of dimension (6+1) and order 3, like the ones in section *10.3*, but corresponding to a different radius. Several examples of this behavior will be provided later on.

A remark on order 4 and this dimension corresponds to the apparent scarcity of Fermat vectors of such an order. For some reason to be elucidated later on, if possible, dimension (4+1) and order 4 present some anomalous properties.

11.3.2. Dimension (7+1)

In this dimension, one has found the following particular examples extracted from a large list of results:

Print ISSN: 2517-276X
Online ISSN: 2517-2778
Website: https://bjmas.org/index.php/bjmas/index
Published by the European Centre for Research Training and Development UK

Order 3: [11, 39, 51, 61, 97, 104, 125] $[r = 164]$

Order 4: [17, 28, 44, 68, 82, 90, 98] $[r = 123]$

Order 5: [5, 17, 103, 171, 180, 224, 239] $[r = 279]$.

Higher orders are quite difficult to find. Such a characteristic is also present in dimensions (8+1) and (9+1).

11.3.3. Natural points in a Fermat surface of dimension (7+1)

Within order 3, one can even choose five true Fermat vectors of dimension (7+1) with the same radius $[r = 42]$, which evidences the existence of Fermat surfaces possessing natural points in this case:

[4, 17, 19, 21, 23, 24, 30]

[5, 8, 19, 21, 22, 27, 30]

[9, 14, 15, 20, 24, 25, 31]

[3, 12, 15, 21, 25, 27, 29]

[3, 4, 18, 20, 24, 28, 29]

11.3.4. Dimension (8+1)

Order 3: [27, 72, 78, 84, 93, 96, 117, 126] $[r = 189]$

Order 4: [46, 68, 80, 106, 204, 218, 230, 241] $[r = 319]$

Order 5: [27, 31, 43, 50, 55, 85, 114, 127] $[r = 142]$

11.3.5. Dimension (9+1)

Order 3: [4, 5, 7, 13, 14, 18, 23, 24, 26] $[r = 38]$

Order 4: [3, 6, 8, 10, 16, 50, 90, 114, 126] $[r = 149]$

Order 5: [3, 23, 25, 31, 43, 48, 51, 52, 76] $[r = 82]$

11.3.6. Natural points in a Fermat surface of dimension (9+1).

Print ISSN: 2517-276X
Online ISSN: 2517-2778
Website: https://bjmas.org/index.php/bjmas/index
Published by the European Centre for Research Training and Development UK

Within order 4, one can even choose 11 true Fermat vectors of dimension (9+1) with the same radius $[r = 149]$, which evidences the existence of Fermat surfaces of such an order possessing natural points:

[3, 6, 8, 10, 16, 50, 90, 114, 126]

[4, 5, 58, 60, 70, 86, 88, 108, 118]

[4, 32, 52, 58, 70, 86, 90, 100, 123]

[4, 16, 32, 40, 66, 68, 80, 119, 120]

[7, 16, 20, 52, 62, 78, 90, 110, 122]

[8, 26, 40, 60, 76, 88, 95, 102, 118]

[10, 14, 18, 52, 54, 86, 94, 100, 125]

[20, 22, 30, 34, 52, 59, 60, 122, 124

[24, 50, 55, 56, 76, 80, 92, 96, 124]

[30, 34, 50, 67, 88, 92, 94, 102, 110]

[48, 52, 60, 64, 74, 82, 95, 100, 118]

Under the same search arrangement, several Fermat vector families with the same radius appear in the same list of dimension (9+1) and order 4. They are not printed to avoid a large list of numerical items.

Such a repeated behavior, similar to the one shown in section 11.3.3, permits us to consider the existence of Fermat surfaces containing natural Fermat vectors with the same Fermat surface radius.

11.3.7. Dimension (10+1)

This already large dimension is not an exception to the behavior of the previous ones. One can show the following obtained vectors:

Order 3: [26, 55, 67, 76, 83, 95, 97, 110, 143, 157] $[r = 225]$

[58, 96, 100, 114, 116, 155, 180, 207, 214, 225] $[r = 355]$

[33, 37, 98, 119, 139, 151, 161, 180, 226, 243]

Order 4: [10, 48, 57, 58, 80, 82, 102, 190, 212, 220] $[r = 277]$

Order 5: [18, 27, 38, 59, 87, 94, 96, 99, 106, 126] $[r = 150]$

In such a dimension for order 3, two Fermat vectors obtained with a radius of 355 are presented, once again proving that Fermat surfaces of such higher dimensions possess natural points.

The order 4 vectors are again very scarce, perhaps pointing to the anomalies related to the number 4, both in dimension and order.

11.4. Discussion on the previous results

With the results shown in sections **9.**, **10.**, and **11.**, one can easily see evidence of the existence of true Fermat vectors in any dimension from (2+1) up to an indefinite (N+1) one.

Vectors of higher orders are difficult to obtain, possibly due to the scarcity of the large natural numbers involved or simply to the nonexistence of natural Fermat surfaces of higher orders.

Nevertheless, the presence of natural Fermat vectors in highly dimensional Minkowski spaces seems sufficient to keep the search for their properties and existence alive.

12. Conclusions

This discussion has presented Fermat's last theorem as the starting point of many research threads within an original general vectorial framework. The main features revealed in the present results involve the following observed items:

I. The theory of natural Fermat vectors can be well-described within natural Minkowski spaces.

II. As a generalization of hyperspheres, one can define Fermat surfaces of any dimension and order that might contain natural Fermat vectors as points on such surfaces.

III. The existence of general true Fermat vectors of order 2 lying on hyperspheres of arbitrary dimension.

IV. True Fermat vectors of order 3 behavior provide the possible proposal of a theorem similar to the original Fermat's last theorem, but in (3+1)-dimensional Minkowski spaces, as well as of the existence of third-order Fermat surfaces in any dimension.

V. The anomalous properties of (4+1)-dimensional natural Fermat vectors with order 5 lead to a further impossible generalization of the Fermat theorem to any dimension and order larger than 3.

VI. Regarding computational difficulty, vector scarcity appears in any dimension of Fermat vector orders higher than 6.

VII. The computational evidence of several natural Fermat vectors of the same dimension, bearing the same radius, indicates the existence of natural vector points in Fermat surfaces.

These seven points suggest that research on true Fermat vectors and surfaces of any dimension and order could be interesting enough to be continued.

Acknowledgments

This paper has benefitted from deep discussions with Prof. Carlos Castro Perelman, Ronin Institute. The author wants to express his gratitude to him for his enlightening remarks.

The encouraging daily help of Blanca Cercas is also wholeheartedly acknowledged.

Finally, the author wants to dedicate this work to Montserrat, his great-granddaughter. She sheds light on hope for every one of us.

Bibliography

1. A. Wiles "Modular Elliptic-Curves and Fermat's Last Theorem". Ann Math **141** (1995) 443-551.
2. S. P. Klykov; "Elementary Proofs for the Fermat Last Theorem in Z Using One Trick for a Restriction in Z_p" J. of Science and Arts 23 (2023) 603-608.
3. S. P. Klykov, M. Klykova; "An Elementary Proof of Fermat's Last Theorem" Research Gate Preprint November (2023) DOI: 10.13140/RG.2.2.19455.59044.
4. J. B. Gilbert; "A Proof of Fermat's Last Theorem" Research Gate Preprint. Revision: July 16th (2023). DOI: 10.13140/RG.2.2.27051.82722
5. R. Carbó-Dorca; "Natural Vector Spaces, (Inward Power and Minkowski Norm of a Natural Vector, Natural Boolean Hypercubes) and Fermat's Last Theorem". J. Math. Chem. **55** (2017) 914-940.
6. R. Carbó-Dorca, C. Muñoz-Caro, A. Niño, S. Reyes; "Refinement of a generalized Fermat's Last Theorem Conjecture in Natural Vector Spaces". J. Math. Chem. **55** (2017) 1869-1877.
7. A. Niño, S. Reyes, R. Carbó-Dorca; "An HPC hybrid parallel approach to the experimental analysis of Fermat's theorem extension to arbitrary dimensions on heterogeneous computer systems". The Journal of Supercomputing **77** (2021) 11328-11352.

8. R. Carbó-Dorca, S. Reyes, A. Niño; "Extension of Fermat's Last Theorem in Minkowski Natural Spaces". J. Math. Chem. **59** (2021) 1851-1863.
9. R. Carbó-Dorca; "Whole Perfect Vectors and Fermat's Last Theorem". Research Gate Preprint. Revision November (2023). DOI: 10.13140/RG.2.2.30320.84484.
10. R. Carbó-Dorca; "Fuzzy sets and Boolean tagged sets, vector semispaces and convex sets, QSM and ASA density functions, diagonal vector spaces and quantum Chemistry". Adv. Molec. Simil. Vol. 2 pg. 43-72. JAI Press, (1998).
11. R. Carbó-Dorca; "Shell partition and metric semispaces: Minkowski norms, root scalar products, distances and cosines of arbitrary order". J. Math. Chem. 32 (2002) 201-223.
12. P. Bultinck, R. Carbó-Dorca; "A mathematical discussion on density and shape functions, vector semispaces and related questions". J. Math. Chem. 36 (2004) 191-200.
13. R. Carbó-Dorca; "Molecular Quantum Similarity Measures in Minkowski Metric Vector Semispaces". J. Math. Chem. 44 (2008) 628-636.
14. R. Carbó-Dorca; "Role of the Structure of Boolean Hypercubes when Used as Vectors in Natural (Boolean) Vector Semispaces". J. Math. Chem. **57** (2019) 697-700
15. R. Carbó-Dorca; "Boolean Hypercubes and the Structure of Vector Spaces". J. Math. Sci. Mod. **1** (2018) 1-14.
16. R. Carbó-Dorca; "Shadows' Hypercube, Vector Spaces, and Non-Linear Optimization of QSPR Procedures". J. Math. Chem. **60** (2022) 283–310.
17. R. Carbó-Dorca, T. Chakraborty; "Extended Minkowski spaces, Zero norms, and Minkowski Hypersurfaces". J. Math. Chem. **59** (2021) 1875-1879.
18. R. Carbó-Dorca; "Generalized Scalar Products in Minkowski Metric Spaces". J. Math. Chem. **59** (2021) 1029-1045.
19. E. Besalú, R. Carbó; "Definition and Quantum Chemical Applications of Nested Summation Symbols and Logical Kronecker Deltas: Pedagogical Artificial Intelligence Devices for Formulae Writing, Sequential Programming, and Automatic Parallel Implementation". J. Math. Chem. **18** (1995)37-72.
20. J. Leech, "Some sphere packings in higher space", Canadian Journal of Mathematics, **16** (1964) 657–682.
21. J. Leech, "Notes on sphere packings", Canadian Journal of Mathematics, **19** (1967) 251–267.

22. Weisstein, Eric W. "Leech Lattice." From *MathWorld*--A Wolfram Web Resource. https://mathworld.wolfram.com/LeechLattice.html