Numerical Approximation of Oscillatory Initial Value Problem Using the Homotopy Analysis Algorithm

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doi: https://doi.org/10.37745/bjmas.2022.0206
Published May 29, 2023

ABSTRACT: In this paper, we present numerical approximation for oscillatory initial value problems (IVPs) using the homotopy analysis algorithm. The convergence of the method is discussed and numerical experiments are presented to illustrate the computational effectiveness of the algorithm. The results obtained are in good agreement with the exact solutions and Adomian decomposition method (ADM). These results show that the algorithm introduced here is accurate and easy to apply without linearization.

KEYWORDS: numerical analysis, initial value problems, differential equations, homotopic analysis

INTRODUCTION

Oscillatory initial value problems arise in mathematical models for problems in mechanics, physics and engineering. These are more difficult to solve analytically, hence it seems more natural to provide direct numerical methods for solving such initial value problems, it is the purpose of this paper to discuss the homotopy analysis algorithm for approximation of oscillatory IVPs. Homotopy Analysis Algorithm (HAA) is a semi-analytic technique used to solve non-linear ordinary differential equations. The Homotopy Analysis Algorithm employs the concept of homotopy in topology to generate a convergent series solution of non-linear systems. This is enabled using a Homotopy-Taylors series to deal with the non-linearity in the system. The Homotopy Analysis Method was proposed by Liao 1992 and the method provides a convenient way to control and adjust the convergence region and rate of series approximation. Ghoreishi et al. (2011) applied HAA to solve a model for HIV infection of CD4+ T-cells. Fallahzadeh and Shakibi (2015) applied homotopy analysis algorithm (HAA) to solve Convection-Diffusion equation. Mkharrhib and Salem (2021) studied new algorithm of the optimal homotopy asymptotic method for solving Lane-Emden equations. Omar (2021a) carried out homotopy analysis-based hybrid genetic algorithm and secent method to solve IVP and higher-order BVP. This work will extend the homotopy analysis algorithm to approximate the solution of oscillatory initial value problems.
Homotopy Analysis Algorithm

Consider the non-linear differential equation

\[ N[U(X)] = 0 \]

(2.1)

\( N \) is a non-linear differential operator, \( X \) denotes the independent variable \( U(X) \) is an unknown function. According to Liao (1992), the zero order deformation equation is given as

\[ [1 - q]L(x; q) - U_0(x)q h N[\Phi(x; q)] \]

(2.2)

Where \( q \in (0; 1) \) is an embedding parameter, \( h \neq 0 \), \( L \) is an auxiliary linear operator \( U_0 \) is the initial guess of \( U(X) \) and \( \Phi(x; q) \) is an unknown function. In this work we assume \( h = -1 \) when \( q = 1 \) in equation (2.2) and \( \phi(x; q) = U_0(x) \)

\( \phi(x; 1) = U(x) \)

Since \( \phi(x; q) \) depends on the parameter \( q \), expanding \( \phi(x; q) \) by Taylor’s series with respect to \( q \)

\[ \phi(x; q) = \phi(x; 0) + \sum_{n=2}^{\infty} U_n(X)q^n \]

\( \Phi(x; q) = U_0 + \sum_{n=1}^{\infty} U_n(X)q^n \)

(2.3)

\[ U_n = \frac{1}{n!} \frac{\partial^n \phi(x; q)}{\partial q^n} | q = 0 \]

(2.4)

From equation (2.2) when \( q = 1 \), we get \( \phi(x; 1)U_0 + \sum_{n=1}^{\infty} U_n(0) \) \( u_n(x) \) can be deduced using the zero-order deformation equation (2.2). Differentiating (2.2) \( n \) times with respect to the embedding parameter \( q \) and \( q=1 \), divide by \( n! \) And get the \( nth \) order deformation equation

\[ L[U_n(X) - X_nU_{n-1}^{n-1} 9X)] = h D_{n-1}[N(\phi(x; q))] \]

(2.5)

Where

\[ D_{n-1}[N(\phi(x; q))] = \frac{1}{(n-1)!} \frac{\partial^{n-1}[N(x; q)]}{\partial q^{n-1}} \]
For example, consider the non-linear differential equation \( \frac{\partial u}{\partial x} - 2u^2 = 0 \) with initial condition \( U_0(0) = 1 \).

\[
L[\phi(x)] = \phi'(x)
\]

\[
N[\phi(X)] = \phi'(X) - 2\phi(X)^2 
\]  
(2.6)

Let \( U_0(0) = 1, \ u_0'(x) = 0 \)

From equation (2.3), we obtain \( u_0 = 0 \) and \( u_n(0) = 0 \ for \ all \ n \geq 1 \)

Let \( \phi = \phi_0(x) + \sum_{n=1}^{\infty} \phi_n q^n = \phi_0 + \phi_1 q + \phi_2 q^2 + \phi_3 q^3 + \ldots \)

\[
= \phi_0(x) + \sum_{n=1}^{\infty} \phi_n q^n = \phi_0 + \phi_1 q + \phi_2 q^2 + \phi_3 q^3 + \ldots 
\]  
(2.7)

Using (2.1), we obtain

\[
I[1 - q] = [L[\phi(X)] - \phi_n(X)] = hq[N[\phi(X)] (1 - q)[\phi - \phi_0] - 2\phi^2]
\]

Substituting (2.5), (2.6) and (2.7) and get

\[
= \phi_0 + \phi_1 q + \phi_2 q^2 + \phi_3 q^3 + \ldots - \phi_0(1 - \beta(\phi_0 + \phi_1 q + \phi_2 q + \phi_3 q^3 + \ldots) 
\]  
(2.8)

Differentiating (2.8) with respect to \( q \) and get the first derivative, to obtain.

\[
(1 - q)[\phi_1 + 2\phi_2 q + 3\phi_3 q^2 + \ldots] + [-I[\phi_1 + \phi_2 q + \phi_3 q^3 + \ldots] = hq[\phi_1 + 2\phi_2 q^3 + 3\phi_3 q^2 + \ldots] 
\]

\[-4[\phi_0 + \phi_1 q + \phi_2 q^2 + \phi_3 q^3 + \ldots] (2.9)
\]

Differentiating (2.1) with respect to \( q \) to get first derivative, and obtain

\[
(1 - q)[\phi_1 + 2\phi_2 q + 3\phi_3 q^2 + \ldots] + [-I[\phi_1 + \phi_2 q + \phi_3 q^3 + \ldots] = hq[\phi_1 + 2\phi_2 q^3 + 3\phi_3 q^2 + \ldots] 
\]

\[-4[\phi_0 + \phi_1 q + \phi_2 q^2 + \phi_3 q^3 + \ldots] (2.10)
\]

\[
+ \phi_2 + 2\phi_3 q + 3\phi_1 q^2 + \ldots]h[\phi_1 + \phi_2 q^3 + \ldots] - 2[\phi_0 + \phi_1 q + \phi_2 q^2 + \phi_3 q^3 + \ldots] (2.11)
\]

When \( q = 0 \) in (2.11) to get

\[
\phi_1 = h[\phi_0 - 2\phi_0^3]
\]
Since \( u_0(x) = 1 \), hence \( \varphi_0(x) = 1 \), then \( \phi_0^1(x) = 0 \), and
\[
\phi_1^1 = h[0 - (L)^2]
\]
\[
\phi_1^1 = -2h
\]
Integrating both sides, to obtain
\[
\phi_1 = -2h + c
\]
(2.12)
since \( u_n(0) = 0, \ V_n \geq 1 \), thus \( \varphi_n(1) = 0, \ V_n \geq 1 \), the \( \varphi_1(0) = 0 \), so \( c = 0 \)

Thus\( \phi_1 = -2hx \) Which is the first derivative of HAA To get the second derivative for (2.12) differentiate the first derivative one time to get
\[
3\phi_3^1 q^1 + \ldots + ( -1)[\phi_1^1 + 2\phi_2^1 q + 3\phi_3^1 q^2 + \ldots] + hq[2\phi_2^1 + 6\phi_3^1 q + \ldots] - 4[\phi_0 + \phi_1 q + \phi_2 q^2 + \phi_3 q^3 + \ldots]
\]
\[
- (2\phi_2 + 6\phi_3 q + \ldots) - 4[\phi_1 + 2\phi_2 q + 3\phi_3 q^2 + \ldots] + h[\phi_1^1 + 2\phi_2^1 q + 3\phi_3^1 q^2 + \ldots] - 4[\phi_0 + \phi_1 q + \phi_2 q^2 + \phi_3 q^3 + \ldots] + [\phi_1^2 + 2\phi_2 q^2 + 3\phi_3 q^3 + \ldots]
\]
(2.12)

Let \( q = 0 \) in (2.12), to get
\[
2\phi_2^1 - \phi_1^1 - \phi_1^1 = h[\phi_1^1 - 4\phi_0\phi_1]
\]
But \( \phi_0 = 1, \phi_1 = -2h \) and then \( \phi_1^1 = -2h + 8h^2x - 2h \)

Integrating both sides, to obtain
\[
\phi_2 = -2h^2x + 4h^2x^2 - 2hx + c \quad \text{Since} \quad \phi_2(0) = 0 \quad \text{then} \quad c = 0 \quad \text{Thus}
\]
\[
\phi_2 = -2h^2x + 4h^2x^2 - 2hx
\]
(2.13)
Which is the second derivative, to get the third derivative for (2.13) differentiate three times to differentiate the second derivative one time.
\[
[1 - q][6\phi_3^1 + \ldots] + [2\phi_2^1 + 6\phi_3^1 q + \ldots]( -1)[2\phi_2^1 + 6\phi_3^1 q + \ldots](-1)[2\phi_2^1 + 6\phi_3^1 q + \ldots]
\]
Let $q = 0$ in (2.13). $6\phi_0 - 2\phi_1 = h[2\phi_1 - 8\phi_0\phi_2 - 4\phi_0\phi_1] + h[2\phi_1 - 8\phi_0\phi_2 - 4\phi_0\phi_1]$

$6\phi_3 = 6\phi_1 = 3h[2\phi_2 - 8\phi_0\phi_2 - 4\phi_0\phi_1]$

Hence $\phi_3 = \frac{1}{2} h[2\phi_1 - 8\phi_0\phi_1 = 4\phi_0\phi_1]$...

(2.14)

Using $\phi_0 = -1, \phi = -hx, \phi_1 = -2h$

$\phi_2 = -2h^2 x + 4h^2 x^2 - 2h$ and $\phi_1 = -2h^2 + 8h^2 x 2h$

Then $\phi_3 = (-2h^2 + 8h^2)(-2h) = \frac{1}{2} h[2(-2h^2 x - 2h) - 8(1)(-2h^2 x + 4h^2 x^2 = 2hx) - 4(-2hx)^2$ Or

$\phi_3 = \frac{1}{2} h[-4h^2 + 16h^2 x - 4h + 16h^2 x - 32h^2 x + 16hx - 16h^2 x^3] = 2h^2 + 8h^2 x^2 - 2h$

Hence $\phi_3 = -2h[h^2 + 2h + 1] + 16h^2 x(h+1) - 24h^3 x^2$

Integrating both sides, to get $\phi_3 = 2h[h^2 + 2h + 1]x + 8h^2(x + 1) - 8h^3 x^3 + c$

The third approximation for $\phi(x)$ is

$\phi_3 = \phi_0 + \sum_{i=0}^{3} \phi_n$

i.e., $\phi_3 = 1 - 2hx - 2h^2 x + 4h^2 x^2 - 2hx - 2h^2 + 2h + 1 + 8h^2 x^2 (h + 1) - 8h^3 x^3$

(2.15)

For $n^{th}$ approximation of $\phi(x)$
\[ \phi_n = \phi_0 + \phi_1 + \phi_2 + \phi_3 + \ldots + \phi_n \]

Hence, \[ \phi_n(x) = \phi_0 + \sum_{n=1}^{\infty} \phi_n = \phi_1 + \phi_2 + \phi_3 + \ldots + \phi_n \]

**Convergence Analysis**

Theorem: Suppose that \( u(x) \) is a Banach space with suitable norm \( ||.|| \), say \( ||.||_\infty \), over which the sequence \( \phi_n(x) \) is defined for a prescribed value of \( h \), Assume also that the initial approximation \( \phi_0(x) \) remains inside the domain of the solution \( u(x) \). Taking \( r \in \mathbb{R} \) as a constant, the following statements holds. If there exist some \( r \in [0, 1] \) such that for all \( k \in \mathbb{N} \), we have \( \| \phi_k + (x) \| + r \| \phi_k (x) \| \), then the series solution \( u(x) = k(x)q_k \) converges absolutely at \( q = 1 \) over the domain of \( x \). Proof Indeed, this is a special case of Banach fix point theorem. See Gambari (2014).

**Numerical Examples**

To demonstrate the effectiveness of the algorithm in this study, we consider the following two examples, the HAA computation results are as presented in the table.

**Example 4.1** Consider the fourth order oscillatory initial value problem

\[ y^{iv} = 5y'' - 4y \]

with the initial conditions

\[ y(0) = 1, \ y'(0) = 0, \ y''(0) = 0, \ y'''(0) = 1 \]

Exact solution: \[ 1 + \frac{1}{6} \sin 2x \]

Source: Bataineh (2009)

To solve the equation by Homotopy Analysis Algorithm with the initial approximation

\[ y(x) = y(0) + y'(0)x + y''(0)x^2 + y'''(0)x^3 \]

\[ y(x) = 1 + \frac{1}{6} x^3 \]

And linear operator
\[ L[\phi(x; q)] = \frac{\delta^2 \phi(x; q)}{\delta x^4} \]

With the property
\[ L[c_1 + c_2 + c_3 + c_4] = 0 \]

Where \( c, [c = 1, 2, 3, 4] \) are constants of integration for \( m \geq 1 \), the mth order deformation with initial conditions will be
\[ y_m(0) = 0, y'_m(0) = 0, y''_m(0) = 0, y'''_m(0) = 1 \]

Where \( R_m(y)_{m-1} = y'''_{m-1}(x) + 5x''_{(m-1)}(x) + 4y_{(m-1)}(x) \)

The solution of the mth order deformation for \( m \geq 1 \)
\[ y_m(x) = y_m y_{m-1}(x) + hL^{-1}R_m(y_{m-1}) \]

Hence \( y'(x) = \frac{1}{6}hx^4 + \frac{1}{24}hx^5 + \frac{1}{1260}hx^7 \)
\[ y^2(x) = \frac{1}{6}hx^3 + \frac{1}{6}hx^2x^4 + \frac{1}{24}h^2x^6 + \frac{1}{1260}h_7x^7 + \frac{29}{5040}h^2x^9 + \frac{1}{5230}h^2x^9 + \frac{1}{2494800}h^2x^{11} \]

.. Then the series solution expression can be written in the form
\[ y(x) = y(0) + y'(x) + y''(x) + y - 3(x) + \ldots \ldots \ldots \text{And so forth} \]

, hence the series solution when \( h = -1 \) is
\[ y_1(x) = -\frac{1}{6}x^4 + \frac{1}{24}x^5 + \frac{1}{1260}x^7 \]
\[ y_2(x) = \frac{1}{36}x^6 + \frac{5}{1008}x^7 + \frac{1}{2520}x^8 + \frac{1}{9072}x^9 + \frac{1}{2494800}x^{11} \]
\[ y_3(x) = -\frac{5}{2010}x^8 + \frac{25}{7257}x^9 - \frac{1}{22680}x^{11} + \frac{1}{133056}x^{12} + \frac{1}{25945920}x^{13} + \frac{1}{2043241200}x^{15} \]
\[ y_4(x) = \frac{5}{36028}x^{10} + \frac{25}{1596672}x^{11} + \frac{1}{399168}x^{12} + \frac{15}{1550755}x^{13} + \frac{1}{90810720}x^{14} + \frac{1}{544864320}x^{15} + \frac{1}{8142948000}x^{16} + \frac{1}{2778803250}x^{17} + \frac{1}{47157617300}x^{19} + \ldots \ldots \]

And so forth.
Hence the series solution is

\[
y(x) = 1 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5 + \frac{1}{36}x^5 + \frac{1}{240}x^7 - \frac{1}{480}x^8 - \frac{17}{2597}x^9 + \frac{17}{181144}x^{10} + \ldots
\]

This converges to the ADM solution

**Approximation of Homotopy Analysis Algorithm**

\[
x, 0 \\
1 + (0)^3 \times 0.1666 = 1 \\
x, 0.1 \\
1 + (0.1)^3 \times 0.1666 = 1. \\
x, 0.2 \\
1 + (0.2)^3 \times 0.1666 = 1.0013 \\
x, 0.3 \\
1 + (0.3)^3 \times 0.1666 = 1.0045 \\
x, 0.4 \\
1 + (0.4)^3 \times 0.1666 = 1.0107 \\
x, 0.5 \\
1 + (0.5)^3 \times 0.1666 = 1.0208 \\
x, 0.6 \\
1 + (0.6)^3 \times 0.1666 = 1.0360 \\
x, 0.7 \\
1 + (0.7)^3 \times 0.1666 = 1.0571 \\
x, 0.8 \\
1 + (0.8)^3 \times 0.1666 = 1.0571 \\
x, 0.9 \\
1 + (0.9)^3 \times 0.1666 = 1.1215 \\
x, (1)
\]
1 + (1)^0.1666 = 1.1666

**Approximation of Adomian Decomposition Method.**

X(0)
1 + (0)^0.1666 = 1

X(0.1)
1 + (0.1)^0.1666 = 1.0017

X(0.2)
1 + (0.2)^0.1666 = 1.0067

X(0.3)
1 + (0.3)^0.1666 = 1.0150

X(0.4)
1 + (0.4)^0.1666 = 1.0267

X(0.5)
1 + (0.5)^0.1666 = 1.0417

X(0.6)
1 + (0.6)^0.1666 = 1.0600

X(0.7)
1 + (0.7)^0.1666 = 1.0816

X(0.8)
1 + (0.8)^0.1666 = 1.1066

X(0.9)
1 + (0.9)^0.1666 = 1.1349

X(1)
1 + (1)^0.1666 = 1.6666

**Approximation of the exact solution**

X,0
1+0.1666\sin 0 = 1
X, 0.1
1+0.1666\sin 0.2 = 1.0331
X, 0.2
1+0.1666\sin 0.4 = 1.1501
X, 0.3
1+0.1666\sin 0.6 = 1.0941
X, 0.4
1+0.1666\sin 0.8 = 1.1125
X, 0.5
1+0.1666\sin 1 = 1.1402
X, 0.6
1+0.1666\sin 1.2 = 1.1553
X, 0.7
1+0.1666\sin 1.4 = 1.1642
X, 0.8
1+0.1666\sin 1.6 = 1.1666
X, 0.8
1+0.1666\sin 1.8 = 1.1673
X, 1
1+0.1666\sin 2 = 1.1728
The HAA compares favourably with the ADM and exact solution.

2. Consider the non-linear oscillatory initial value problem

\[ y^4 = yy'' + y^2 \]

Subject to the initial conditions

\[ y(0) = 0, y_1(0) = 1, y_2(0), y_3(0) = 1 \]

Source: Liao (2012)

The exact solution is \( e^x - 1 \), According to the Homotopy Analysis Algorithm, the initial approximation is \( y_0(x) = x + \frac{1}{2} x^2 + \frac{1}{6} x^3 \)
The zero order deformation equation with initial conditions

\[ Rm_{(ym-1)} = y''_{m-1} + \sum_{m=0}^{n-1} y'(x)y''_{m-1}(0) - yf(x)_{m-1} \sum_{j=0}^{m-1} y^j - j(x) \]

The solution of the mth order deformation equation for \( m \geq 1 \) is

\[ y_m(x) = y_m y_{m-1}(x) + hL - 1 R_m y_{m-1} y(x) = -\frac{1}{24h^4} - \frac{1}{120} hx^5 - \frac{1}{270} hx^6 - \frac{1}{5040} hx^7 + \frac{1}{40320} hx^8 + \ldots \]

The series solution expression can be written in the form \( y(x) = y(x) + y'(x) + y''(x) + \ldots \) and so forth.

This converges to the Adomian Decomposition Method solution

**Approximation of Homotopy Analysis Algorithm**

\[ X,(0) \]
\[ 0 + (0)^2 = 0 \]
\[ X (0.1) \]
\[ 0.1 + 0.5(0.1)^2 = 0.1050 \]
\[ X (0.2) \]
\[ 0.2 + 0.5(0.2)^2 = 0.2100 \]
\[ X ,(0.3) \]
\[ 0.3 + 0.5(0.3)^2 = 0.3450 \]
\[ X,(0.4) \]
\[ 0.4 + 0.5(0.4)^2 = 0.4850 \]
\[ X,(0.5) \]
\[ 0.5 + 0.5(0.5)^2 = 0.6250 \]
\[ X,(0.6) \]
\[ 0.6 + 0.5(0.6)^2 = 0.7800 \]
\[ X,(0.7) \]
\[ 0.7 + 0.5(0.7)^2 = 0.9450 \]
\[ X,(0.8) \]
\[ 0.8 + 0.5(0.8)^2 = 1.1200 \]

\[ X,(0.9) \]
\[ 0.9 + 0.5(0.9)^2 = 1.3050 \]

\[ X, 1 \]
\[ 1 + 0.5(1)^2 = 1.5000 \]

**Approximation of Adomian Decomposition Method**

\[ X,(0) \]
\[ 0 + 0.5(0)^3 = 0 \]

\[ X,(0.1) \]
\[ 0.1 + 0.5(0.1)^3 = 0.1005 \]
\[ 0.2 + 0.5(0.2)^3 = 0.2040 \]

\[ X,(0.3) \]
\[ 0.3 + 0.5(0.3)^3 = 0.3135 \]

\[ X,(0.4) \]
\[ 0.4 + 0.5(0.4)^3 = 0.4320 \]

\[ X,(0.5) \]
\[ 0.5 + 0.5(0.5)^3 = 0.5625 \]

\[ X,(0.6) \]
\[ 0.6 + 0.5(0.6)^3 = 0.7080 \]

\[ X,(0.7) \]
\[ 0.7 + 0.5(0.7)^3 = 0.8715 \]

\[ X,(0.8) \]
\[ 0.8 + 0.5(0.8)^3 = 1.0560 \]

\[ X,(0.9) \]
\[ 0.9 + 0.5(0.9)^3 = 1.2645 \]
X,(1)
1+0.5(1)=1.5000

Approximation of the exact solution

$e^{0.1-1}=0.1051$
$e^{0.2-1}=0.2214$
$e^{0.3-1}=0.3499$
$e^{0.4-1}=0.4918$
$e^{0.5-1}=0.6487$
$e^{0.6-1}=0.8221$
$e^{0.7-1}=1.0137$
$e^{0.8-1}=1.2256$
$e^{0.9-1}=1.4596$
$e^{1-1}=1.7183$

Table (4.2)

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<th>X</th>
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<th>ADM</th>
<th>EXACT</th>
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<th>ADM ERROR</th>
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<td>1.5000</td>
<td>1.7183</td>
<td>0.2183</td>
<td>0.2183</td>
</tr>
</tbody>
</table>

CONCLUSION

The proposed algorithm HAA have been successfully applied for the approximation of
oscillatory initial value problems. The result obtained is compared with the Adomian Decomposition Method and the exact solution, it was observed that all the problems considered shows that the HAA results compared favourably with the ADM and exact solutions, It is clearly seen that the Homotopy Analysis Algorithm is a cogent and effective algorithm for approximating the numerical (analytic) solution of oscillatory initial value problems, also It could be observed that HAA converges faster and was implemented without any need for discretization of the problem, Therefore for easy of solution to oscillatory IVPs, without tedious algebraic computations, this study recommends HAA for approximating oscillatory IVPs.

REFERENCES
11. HE J.H (2003), Homotopy perturbation method, a new nonlinear analytical technique- Applied mathematics and computation No. 125 (73-79)