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The Third Romberg Extrapolate as a Numerical Integration

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Abstract: *Modern techniques for quadrature include, but are not limited to, the Trapezium rule, the Midpoint rule, and Simpson's rule. The accuracy of these methods can be improved by employing Romberg's method. This is achieved by applying each method to a definite integral, subdividing it into multiple intervals, and then taking appropriate linear combinations of the resulting estimates to produce approximations with high-order accuracy. In this work, the third Romberg extrapolate is applied to a definite integral, and its solution is compared with the exact solution to demonstrate its accuracy.*

Keywords: Romberg method, definite integrals, Trapezium rules, quadratures

INTRODUCTION

Numerical integration is a method used to approximate the definite integral of a function $f(x)$ over the interval [a, b], which represents the signed area under the curve of the function between the limits a and b.

These methods are referred to as quadrature and one of such methods is the Romberg method. It is a method that is based on the trapezoidal rule and the Richardson extrapolation. This works by using results from the trapezoidal rule and at smaller step sized and these results are extrapolates in order to improve its accuracy [1].

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The Romberg method is widely used for integrals without analytical solutions and have been applied by engineers, physicists and mathematicians.

Over the years, researchers have applied, explored, and combined Romberg methods with other methods because of its computational efficiency. One of such is the study by [2], which explores extrapolation quadrature from equispaced samples of functions with jumps, highlighting the effectiveness of extrapolation techniques in improving accuracy.

In [3], the authors compared the Clenchaw-Curtis quadrature (introduced in [4]) with the Romberg method to determine the quadrature with the superior accuracy. In 1997, Cools reviewed various multidimensional integration methods, including the Romberg integration, and other extrapolation techniques focusing on the application of Romberg method in higher-dimensional problems [5].

Feldmann implemented the Romberg integration on parallel computing architectures and also discussed how to optimize it for high-performance computing in order to make it suitable for large-scale computational problems [6].

Other authors have applied the Romberg methods in atmospheric modelling, electromagnetic fields, quantum chemistry, plasma physics, computational biology [7-12].

As a remark, it is nice to note that the Third Romberg Extrapolate, as a numerical integration technique, can be effectively employed in conjunction with semi-analytical methods, such as the Adomian Decomposition Method (ADM) and Homotopy Perturbation Method (HPM), to enhance the accuracy of integral evaluations within the iterative solution process [13-17]. This approach is particularly beneficial for complex differential models where analytical integration is challenging, offering improved convergence rates and precision, especially when dealing with nonlinear terms or boundary conditions [18-22].

The definite integral to be considered in this work is of the form:

$$I = \int_a^b f(x)\,dx \tag{1}$$

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METHODOLOGY AND BASIC THEOREM

Theorem (See [23] for proof)

If all derivatives of $f(x)$ exist and are continuous on [a, b], then the composite trapezium rule may be expressed in the form:

$$\int_a^b f(x)dx = \frac{h}{2}\left[f_0 + 2\sum_{r=1}^{n-1} f_r + f_n \right] + \sum_{l=1}^{\infty} \alpha_l h^{2l} \tag{2}$$

where $h = \frac{b-a}{n}$, $a = x_0$, $b = x_n$, $x_k = x_0 + kh$, $(k = 1, 2, \ldots, n-1)$ and α_l $(l = 1, 2, \ldots)$ are independent of h for sufficiently small h.

Romberg Extrapolation

Let $T_{k,0}$ $(k = 1, 2, \ldots)$ denote the trapezium rule estimate of I (2) using 2^k sub-intervals of length $h_k = \frac{b-a}{2^k}$.

From the above theorem;

$$I = T_{k,0} + \alpha_1 h_k^2 + \alpha_2 h_k^4 + \alpha_3 h_k^6 + \cdots \tag{3}$$

If we now half the value of h, and double the number of subintervals; (3) becomes

$$I = T_{k+1,0} + \alpha_1 \left(\frac{h_k}{2}\right)^2 + \alpha_2 \left(\frac{h_k}{2}\right)^4 + \alpha_3 \left(\frac{h_k}{2}\right)^6 + \cdots$$

$$= T_{k+1,0} + \frac{1}{4}\alpha_1 h_k^2 + \frac{1}{16}\alpha_2 h_k^4 + \frac{1}{64}\alpha_3 h_k^6 + \cdots \tag{4}$$

To eliminate terms involving h_k^2, multiply (4) by 4 then subtract (3) from the result to get:

$$3I = 4T_{k+1,0} - T_{k,0} - \frac{3}{4}\alpha_2 h_k^4 - \frac{15}{16}\alpha_3 h_k^6 + \cdots \tag{5}$$

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$$\Rightarrow \quad I = \frac{4T_{k+1,0} - T_{k,0}}{3} - \frac{1}{4}\alpha_2 h_k^4 - \frac{5}{16}\alpha_3 h_k^6 + \cdots \tag{6}$$

Or

$$I = T_{k+1,1} - \frac{1}{4}\alpha_2 h_k^4 - \frac{5}{16}\alpha_3 h_k^6 + \cdots \tag{7}$$

where $T_{k+1,1} = \dfrac{4T_{k+1,0} - T_{k,0}}{3}$ (8)

The quantity, $T_{k+1,1}$ in (8) is called the first Romberg extrapolate and is of order 4.

We repeat the extrapolation process from (7) which yield

$$I = T_{k,1} + \beta_1 h_k^4 + \beta_2 h_k^6 + \cdots \tag{9}$$

for some β_l independent of h.

Again, we halve the value of h and hence double the number of sub-intervals to have

$$I = T_{k+1,1} + \beta_1 \left(\frac{h_k}{2}\right)^4 + \beta_2 \left(\frac{h_k}{2}\right)^6 + \cdots$$

$$= T_{k+1,1} + \frac{1}{16}\beta_1 h_k^4 + \frac{1}{64}\beta_2 h_k^6 + \cdots \tag{10}$$

To eliminate β_1, multiply (10) by 16 then subtract (9) from the result to get:

$$(10)\times 16 \quad \Rightarrow \quad 16I = 16T_{k+1,1} + \beta_1 h_k^4 + \frac{1}{4}\beta_2 h_k^6 + \cdots \tag{11}$$

$$(11)-(9) \quad \Rightarrow \quad 15I = 16T_{k+1,1} - T_{k,1} - \frac{3}{4}\beta_2 h_k^6 + \cdots \tag{12}$$

$$\Rightarrow \quad I = \frac{16T_{k+1,1} - T_{k,1}}{15} - \frac{1}{20}\beta_3 h_k^6 + \cdots \tag{13}$$

Or

$$I = T_{k+1,2} - \frac{1}{20}\beta_2 h_k^6 + \cdots \tag{14}$$

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where $T_{k+1,2} = \dfrac{16T_{k+1,1} - T_{k,1}}{15}$ (15)

The quantity, $T_{k+1,2}$ in (15) is called the second Romberg extrapolate and is of order 6.

The process can be continued and in general, we have

$$I = T_{k,l} + \lambda_1 h_k^{2l+2} + \lambda_2 h_k^{2l+4} + \cdots \tag{16}$$

Again, we halve the value of h and hence double the number of sub-intervals to have

$$I = T_{k+1,l} + \lambda_1 \left(\frac{h_k}{2}\right)^{2l+2} + \lambda_2 \left(\frac{h_k}{2}\right)^{2l+4} + \cdots \tag{17}$$

Multiply (17) by 2^{2l+2} then subtract (16) from the result to get:

$$(17) \times 2^{2l+2}: \quad 2^{2l+2} I = 2^{2l+2} T_{k+1,l} + \lambda_1 h_k^{2l+2} + 2^{2l-2} \lambda_2 h_k^{2+4} + \cdots \tag{18}$$

$$(18) - (16): \quad \left(4^{l+1} - 1\right) I = 4^{l+1} T_{k+1,l} - T_{k,l} - \frac{3}{4} \lambda_2 h_k^{2l+4} + \cdots \tag{19}$$

$$\Rightarrow \quad I = \frac{4^{l+1} T_{k+1,l} - T_{k,l}}{4^{l+1} - 1} + O\left(h^{2l+4}\right) \tag{20}$$

Or $$I = T_{k+1,l+1} + O\left(h^{2l+4}\right) \tag{21}$$

Where $T_{k+1,l+1} = \dfrac{4^{l+1} T_{k+1,l} - T_{k,l}}{4^{l+1} - 1}$ (22)

The quantity $T_{k+1,l+1}$ is the generalized Romberg extrapolate and is of order $2l + 4$ $(l = 1, 2, \ldots)$ That is $O\left(h^{2l+4}\right)$.

The Romberg extrapolates are usually set out in triangular array of the form:

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$T_{0,0}$					
$T_{1,0}$	$T_{1,1}$				
$T_{2,0}$	$T_{2,1}$	$T_{2,2}$			
$T_{3,0}$	$T_{3,1}$	$T_{3,2}$	$T_{3,3}$		
…	…	…	…	…	
…	…	…	…	…	…

RESULT AND DISCUSSION

In this section, the third Romberg extapolate is used as a numerical integration of an exponential function.

Example 1: To find the third Romberg extrapolate of the definite integral:

$$\int_0^2 e^x dx, \qquad (23)$$

the solution is presented as follows:

For the initial values, $T_{0,0}, T_{1,0}, T_{2,0}, T_{3,0}$ using the trapezium rule we have:

$$T_{0,0} = \frac{2}{2}\left(e^0 + e^0\right) = 1 + e^2 = 8.3890561$$

$$T_{1,0} = \frac{1}{2}\left(e^0 + 2e^1 + e^2\right) = 6.9128099$$

$$T_{2,0} = \frac{0.5}{2}\left(e^0 + 2\left(e^{0.5} + e^1 + e^{1.5}\right) + e^2\right) = 6.5216101$$

$$T_{3,0} = \frac{0.25}{2}\left(e^0 + 2\left(e^{0.25} + e^{0.5} + e^{0.75} + e^1 + e^{1.25} + e^{1.5} + e^{1.75}\right) + e^2\right) = 6.4222978$$

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Next apply the first Romberg extrapolate:

$$T_{k+1,1} = \frac{4T_{k+1,0} - T_{k,0}}{3}$$

$$T_{1,1} = \frac{4T_{1,0} - T_{0,0}}{3} = \frac{4(6.9128099) - 8.3890561}{3} = 6.4207278$$

$$T_{2,1} = \frac{4T_{2,0} - T_{1,0}}{3} = \frac{4(6.5216101) - 6.9128099}{3} = 6.3912102$$

$$T_{3,1} = \frac{4T_{3,0} - T_{2,0}}{3} = \frac{4(6.4222978) - 6.5216101}{3} = 6.3891937$$

Again apply the second Romberg extrapolate:

$$T_{k+1,2} = \frac{16T_{k+1,1} - T_{k,1}}{15}$$

$$T_{2,2} = \frac{16T_{2,1} - T_{1,1}}{15} = \frac{16(6.3912102) - 6.4207278}{15} = 6.3892424$$

$$T_{3,2} = \frac{16T_{3,1} - T_{2,1}}{15} = \frac{16(6.3891937) - 6.3912102}{15} = 6.3890593$$

Finally, the third Romberg extrapolate is given as:

$$T_{k+1,3} = \frac{4^3 T_{k+1,2} - T_{k,2}}{4^3 - 1} = \frac{64T_{k+1,2} - T_{k,2}}{63}$$

$$T_{3,3} = \frac{64T_{3,2} - T_{2,2}}{63} = \frac{64(6.3890593) - 6.3892424}{63} = 6.3890564$$

Exact Solution: $\int_0^2 e^x dx = e^x \Big|_0^2 = e^2 - e^0 = e^2 - 1 = 7.3890561 - 1 = 6.3890561$

Hence, error the is 0.0000003.

CONCLUDING REMARKS

In this work, the third Romberg extrapolate was applied to a definite integral, and the solution was compared with the exact solution to demonstrate its accuracy. The Romberg method, which is based on the trapezoidal rule and Richardson extrapolation, enhances the precision of numerical integration by utilizing results from the trapezoidal rule at progressively smaller step sizes. These results are then extrapolated to obtain more accurate approximations, highlighting the effectiveness of the Romberg method in improving accuracy for definite integrals.

Further research could explore the application of Romberg extrapolation to more complex integrals, such as those involving higher-dimensional spaces or integrands with singularities. Additionally, integrating the Romberg method with other numerical techniques or semi-analytical methods could enhance its applicability to solving differential equations. Investigating the efficiency of the Romberg method in adaptive quadrature schemes and real-world engineering or physics problems where high precision is required would also provide valuable insights.

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